## *SOLUTIONS OF PROBLEMS IN NUMBER THREE.*

---

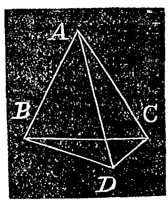
SOLUTIONS of problems in No. 3 have been received as follows:

From J. M. Boorman, 393, 394; Prof. W. P. Casey, 399; Geo. E. Curtis, 395; George Eastwood, 392; William Hoover, 398; Henry Heaton, 399; Prof. Asaph Hall, 397, 400; Thos. Spencer, 397, 399; M. Updegraff, 399; Prof. C. Van Velzer, 397.

---

392. *By George Eastwood, Saxonville, Mass*:—"In a triangular pile of round shot, each shot rests upon three other shot, thus forming an empty space. It is required to find the ratio of the capacity of all the spaces to the capacity of all the balls."

SOLUTION BY THE PROPOSER.




In the annexed diagam, let $A$ represent the centre of the top shot, and $B$, $C$, $D$ the centres of the shot on the second course from the top of the pile. Conceive a plane to pass through these centres, and to become the base of a regular tetrahedron whose vertex is $A$. For the edges we have, suppose, $AB = AC = AD = BC = BD = CD = 2r$.

The angles $BAC$, $CAD$, $DAB$ are evidently equal; hence their respective cosines are equal, and each equal to $\frac{1}{2}$. And from an inspection of the figure, we readily perceive that each solid cuts from its concentric shot, a spherical triangle which forms the base of a spherical pyramid that is common to the tetrahedron and said shot. But the sides of the spherical triangles are obviously the measures of the angles $BAC$, $CAD$, $DAB$, whose respective cosines we have already found to be each $= \frac{1}{2}$.

Let the spherical excess of each spherical triangle be designated by $\varepsilon$, and its area by $\Sigma$. By a formula due to *De Gua*, we have

$$\operatorname{vers}\varepsilon = \frac{1-\cos^2 A_1-\cos^2 A_2-\cos^2 A_3+2\cos A_1\cos A_2\cos A_3}{(1+\cos A_1)(1+\cos A_2)(1+\cos A_3)}$$

$$= \frac{4}{27}.$$

Whence $\varepsilon = \operatorname{vers}^{-1}\frac{4}{27} = \cos^{-1}\frac{23}{27}$.

From this value of $\varepsilon$ we have $\Sigma = r^2\varepsilon\dfrac{3.1416}{180^\circ} = r^2\cos^{-1}\dfrac{23}{27}\dfrac{3.1416}{180^\circ}$.

The capacity of each spherical pyramid is $\frac{1}{3}r\Sigma$, and the capacity of the four is $\frac{4}{3}r\Sigma$.

By a formula of Le Gendre, the capacity, $k$, of the tetrahedron is

$$k = \frac{AB.AC.AD\sqrt{(1-\cos^2 A_1-\cos^2 A_2-\cos^2 A_3+2\cos A_1\cos A_2\cos A_3)}}{6}$$

$$= \tfrac{2}{3}r^3\sqrt{2}.$$

Hence the capacity of *one* empty space is

$$\frac{r^3}{3}\left(2\sqrt{2}-4\Sigma\right) = \frac{r^3}{3}\left(2\sqrt{2}-4\cos^{-1}\frac{23}{27}\frac{3.1416}{180^\circ}\right)$$

$= 0.20772r^3 =$ capacity of *one* shot $\times .0496$ f

$=$ capacity of one shot $\div$ 20.16.

Let $n =$ number of shot in one side of the bottom course of the pile, then

$\frac{1}{6}n(n+1)(n+2) =$ number of shot in the whole pile, and

$\frac{1}{6}n(n-1)(n+1) =$ number of empty spaces in same.

Hence, capac. of whole pile : capac. of empty sp's :: $(20.16)(n+1):(n-1)$.

---

393. *By J. M. Boorman, Esq., New York City.*—State the general equation of the 4th degree in terms whose coefficients shall be real and direct functions of its roots and admit a solution showing the root's *real* nature—no root to be directly *expressed* by a letter.

394. *Id.*— Show that the general equation of the 4th degree has its companion biquadrate, and state it and the respective relat'ns of their roots.

ANSWER BY THE PROPOSER.

$$X^4-4dX^3+(6d^2-2a)X^2-(4d^3-4ad-f)X+d^4-2ad^2-fd+a^2-b=0, \quad (A)$$

which, for $d=0$, or by removing the second term, is

$$X^4 - 2aX^2 + fX + a^2 - b = 0; \quad (B)$$

where $X = d \,{}_{\backslash}\!\pm \tfrac{1}{2}S \pm\sqrt{[a + FS \,{}_{\backslash}\!\mp F(f \text{ and } S)]}$, (C)

$S = r+t;\ r+l;\ r+e;\ t+e;\ r, t, l, e$ being the 4 roots of (B), and $a = \frac{1}{4}(S^2+S_{\backslash}^{\,2}+S_{\backslash\backslash}^{\,2})$; $b = \frac{1}{4}(S^2S_{\backslash}^{\,2}+S^2S_{\backslash\backslash}^{\,2}+S_{\backslash}^{\,2}S_{\backslash\backslash}^{\,2})$; $f = SS_{\backslash}S_{\backslash\backslash}$; $S, S_{\backslash}, S_{\backslash\backslash}$ being 3 roots of the equation $S^6-4aS^4+4bS^2-f^2 = 0$.

The inverted prime (${}_{\backslash}$), above, means that $\pm$ signs so marked change only in unison.

[In common with some of our readers, we failed to perceive the exact sig-significance of these questions, but, as the author claims to have made some important discoveries in relation to the roots of such equations, we hoped to be enlightened by his answers. The foregoing answer has been submitted and is accompanied by an Example, and also an answer to 394, but as we fail to understand equation (*C*) above we withhold the remainder until that equation shall be more explicitly stated.—Ed.]

395. *By C. O. Boije af Gennas, Gothenburg, Sweden.*—"Determine the law of density of a sphere in order that its centre of gravity may be coincident with the centre of gravity of the half sphere cut off from the sphere."

SOLUTION BY GEO. E. CURTIS, NEWHAVEN, CONN.

The problem is essentially indeterminate. A simple law is obtained as follows. The distance of the center of gravity of a sphere from a point on its circumference taken as the pole is, by a formula of polar coordinates,

$$\frac{\int_0^{\frac{1}{2}\pi}\int_0^{2a\cos\theta} D.r^3\sin\theta\cos\theta\,d\theta\,dr}{\int_0^{\frac{1}{2}\pi}\int_0^{2a\cos\theta} D.r^2\sin\theta\,d\theta\,dr},$$

where $a$ is the radius and $D$, the density.

The center of gravity must coincide with the center of gravity of a homogeneous hemisphere of equal radius; hence the integral must equal $\frac{5}{8}a$.

If the law of density is $1\div(r\cos\theta)$, the integral gives the distance $\frac{2}{3}a^3\div a^2$; if the law is $1\div r$ we obtain $\frac{8}{15}a^3\div\frac{2}{3}a^2$. As $1\div(r\cos\theta)$ gives a result slightly too large, subtract from it $1\div r$ multiplied by a constant factor, $c$, and we shall obtain the required law.

To obtain the constant factor, we have the equation $(\frac{2}{3}-\frac{8}{15}c)\div(1-\frac{2}{3}c)=\frac{5}{8}$. This gives $c=\frac{5}{14}$, and the corrected law of the density is

$$\frac{1}{r\cos\theta}-\frac{5}{14r}.$$

---

396. No solution received.

397. *By Prof. J. M. Rice, U. S. Naval Academy.*—"Given

$$\varphi(x^2)\,\varphi(y^2) = \varphi(x'^2)\,\varphi(y'^2)$$

and

$$x^2+y^2 = x'^2+y'^2,$$

to determine the form of the function denoted by $\varphi$."

SOLUTION BY PROF. ASAPH HALL.

Put $x^2=u$, $y^2=v$, etc., then

$$\varphi(u)\,.\,\varphi(v) = \varphi(u')\,.\,\varphi(v') = f(u,v).$$

Differentiating,

$$\varphi'(u)\,.\,\varphi(v) = f'(u,v): \qquad \text{and } \varphi(u)\,.\,\varphi'(v) = f'(u,v).$$

Hence

$$\varphi'(u)\,.\,\varphi(v) = \varphi(u)\,.\,\varphi'(v), \text{ or}$$

$$\frac{\varphi'(u)}{\varphi(u)} = \frac{\varphi'(v)}{\varphi(v)} = k; \text{ a constant.}$$

Therefore $\log \varphi(u) = C.ku,$

and $\varphi(x^2) = C.e^{kx^2}.$

The second given equation does not seem to be necessary for the solution, but follows as a result.

SOLUTION BY THOMAS SPENCER, SOUTH MERIDEN, CONN.

Take the logarithm of the first equation to any base $a$, and we have

$$\log_a\varphi(x^2) + \log_a\varphi(y^2) = \log_a\varphi(x'^2) + \log_a\varphi(y'^2).$$

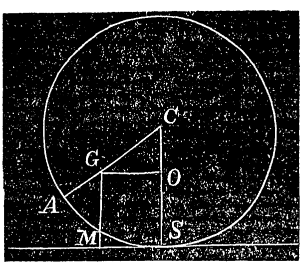
But $x^2+y^2 = x'^2+y'^2$ is of that form; therefore we are at liberty to put $\log_a\varphi(x^2) = bx^2$, $\log_a\varphi(y^2) = by^2$, $\log_a\varphi(x'^2) = bx'^2$ and $\log_a\varphi(y'^2) = by'^2$, where $b$ is any quantity.

Hence we have $\varphi(x^2) = a^{bx^2}$, $\varphi(y^2) = a^{by^2}$, &c., which gives the form of the function.

---

398. *By Wm. Hoover, A. M., Dayton, Ohio*—"An angular velocity having been impressed upon a heterogeneous sphere, about an axis, perp. to the vertical plane which contains its center of gravity $G$ and geometrical center $C$, and passing through $G$, it is then placed on a smooth horizontal plane; to find the magnitude of the impressed angular velocity that $G$ may rise into a point in the vertical line $SCK$ through $C$, and there rest; the angle $GCS$ being $a$ at the beginning of the motion, $c$, the radius, and $\varphi$ the req'd angular velocity."

SOLUTION BY THE PROPOSER.




Draw the radius $CGA$, and from $G$ drop the perpendicular $GM$ upon the plane.

Let $m$ = the mass of the sphere, $k$, the radius of gyration of the sphere about an axis through $G$ perpendicular to the plane containing $C$ and $G$, $R$, the mutual reaction of the sphere and the plane, $SM = x$, $GM = y$, $CS = CA = a$, ang. $AGM$ = angle $ACS = \varphi$ and $CG = c$.

Since there is no friction, we have for the motion of the sphere

$$m\frac{d^2x}{dt^2} = 0; \qquad (1)$$

resolving forces vertically,

$$m\frac{d^2y}{dt^2} = R - mg, \qquad (2)$$

and taking moments about $G$,

$$mk^2 \frac{d^2\varphi}{dt^2} = -Rc\sin\varphi, \tag{3}$$

$\varphi$ being the angular velocity of the sphere.

We have from the geometry, $y = ac\cos\varphi$, whence

$$\frac{d^2y}{dt^2} = c\sin\varphi\frac{d^2\varphi}{dt^2} + c\cos\varphi\frac{d\varphi^2}{dt^2}.$$

Substituting in (2),

$$R = m\left(c\sin\varphi\frac{d^2\varphi}{dt^2} + c\cos\varphi\frac{d\varphi^2}{dt^2} + g\right).$$

This in (3) gives by reduction

$$(c^2\sin^2\varphi + k^2)\frac{d^2\varphi}{dt^2} + c^2\sin\varphi\cos\varphi\frac{d\varphi^2}{dt^2} = -cg\sin\varphi. \tag{4}$$

Integrating,

$$(c^2\sin^2\varphi + k^2)\frac{d\varphi^2}{dt^2} = C + 2cg\cos\varphi. \tag{5}$$

Let $t = 0$, when $\varphi = 0$; $d\varphi \div dt = \omega$, and $C = (c^2\sin^2\varphi + k^2)\omega^2 - 2cg$. Hence (5) becomes

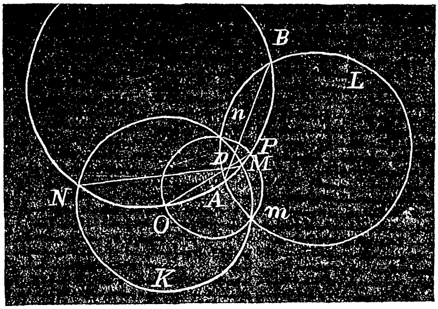
$$(c^2\sin^2\varphi + k^2)\frac{d\varphi^2}{dt^2} = (c^2\sin^2\varphi + k^2)\omega^2 - 2cg(1 - \cos\varphi). \tag{6}$$

Now if the initial value of $\varphi = \alpha$, the terminal value $= \alpha + \pi$, when also $\frac{d\varphi}{dt} = 0$; then the left member of (6) becomes 0, and $\omega^2 = \frac{2cg(1-\cos\varphi)}{c^2\sin^2\varphi + k^2}$.

---

399. *By E. J. Esselstyn, New Haven, Conn.*—Given two points $A$ and $B$, and a circle $K$ having its centre at $O$. Let any circle $L$ be drawn thro' $A$ and $B$ so as to cut the circumf. of the circle $K$ in two variable points $m$ and $n$. Show that the circle through $O$, $A$ and $B$ is cut by the variable circle through $O$, $m$ and $n$, in a fixed point $P$.

SOLUTION BY HENRY HEATON.




In the figure the circle through $A$, $B$ and $O$ intersects the circle $K$ in $M$ and $N$, and the chords $AB$ and $MN$ intersect in $D$; $\therefore AD \times DB = MD \times DN$. If we draw $MD$ and prolong it till it cuts the circle $K$ in $n'$ and the circle $L$ in $n''$, we have $mD \times Dn' = MD \times DN$, and $mD \times Dn'' = AD \times DB$. Hence $Dn' = Dn''$, or $n'$ and $n''$ coincide in the point $n$.

If we draw $OD$ and prolong it until it cuts the circle $OAB$ in $P'$ and the circle $Omn$ in $P''$, we have $OD \times DP' = AD \times DB$ and $OD \times DP'' = mD \times Dn = AD \times DB$. Hence $DP' = DP''$ or $P'$ and $P''$ coincide in the point $P$. But $AB$ and $MN$ being fixed lines $D$ is a fixed point, and the line $OD$ cuts the circle $OAB$ in the fixed point $P$.

SOLUTION BY THOMAS SPENCER.

$M$, $N$ being the intersections of the circles $K$ and $OAB$, we see that the straight lines $MN$, $AB$, and $mn$ are the radical axes of the three circles $K$, $L$ and $OAB$, and $D$ their radical centre, which is a fixed point, because the lines $MN$ and $AB$ are fixed. Also $D$ is the radical centre of the three circles $K$, $Omn$ and $OAB$, $mn$, $MN$ and $OP$ being their radical axes; $\therefore$ $OP$ is a fixed straight line, because the points $O$ and $D$ are fixed points. Hence, because the circle $OAB$ is invariable, the point $P$ is fixed.

---

400. *By Prof. Asaph Hall.*—"In a plane passing through the centre of the sun, 12 right lines are drawn from this centre making an angle of 30° with each other. On each of these lines, three homogeneous spherical bodies are placed at distances respectively of 10, 20 and 30 from the centre of the sun; the distance from the earth to the sun being the unit of distance.

The mass of each of these bodies being equal to that of the sun, what will be the velocity of a particle that starts from an infinite distance and moves in a right line towards the centre of the sun, and perpendicular to the plane of the bodies, when the particle is at a distance of 0.01 from the centre of the sun; the law of attraction being that of Newton?"

SOLUTION BY PROFESSOR HALL.

Let $m$ be the mass of the sun, $x$ the distance of the particle from the centre of the sun at the time $t$; then the equation of motion is

$$\frac{d^2x}{dt^2} + m.\left\{\frac{1}{x^2} + \frac{12x}{(x^2+100)^{\frac{3}{2}}} + \frac{12x}{(x^2+400)^{\frac{3}{2}}} + \frac{12x}{(x^2+900)^{\frac{3}{2}}}\right\} = 0.$$

Integrating,

$$\frac{dx}{dt} = v = (2m)^{\frac{1}{2}}\left\{\frac{1}{x} + \frac{12}{(x^2+100)^{\frac{1}{2}}} + \frac{12}{(x^2+400)^{\frac{1}{2}}} + \frac{12}{(x^2+900)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.$$

Let $v'$ be the mean velocity of the earth in its orbit at the distance unity, then $v'^2 = m$: and $v' = 18.4166$ miles per second. Hence $v = 263.3$ m's per second. If there were no body but the sun, the velocity at the same distance would be 260.5 miles per second. The addition of the other bodies therefore produces only a small change in the velocity. An increase in the mass of the sun would be much more effective.